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Ising transition in the classical ANNNXY model

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Abstract. We investigate the XY model with ferromagnetic nearest and antiferromagnetic next-to-nearest neighbours couplings (ANNNXY model). We formulate the study of both phases of the model in terms of different Coulomb gas models. The non-frustrated ferromagnetic phase is thus transformed into a usual Coulomb gas with one species, whereas the frustrated phase is translated into a Coulomb gas with three species interacting with each other in an anisotropic lattice. We then generalize the Kosterlitz–Thouless renormalization group equations for both phases by treating the Ising and Kosterlitz–Thouless order parameters in an independent way. This enables us to discuss the nature of the transition (Ising and (or) Kosterlitz–Thouless) in the frustrated phase.

1. Introduction

Since the discovery of high-temperature superconductivity, a considerable interest has emerged in two-dimensional frustrated antiferromagnetic systems. In particular adding frustration could be a way to describe how the now widely accepted long-range order in two-dimensional Heisenberg antiferromagnets could be destroyed, leading to a spin-liquid state [1–3]. One way to induce frustration is to add competing interactions between nearest neighbours (NN) and next-to-nearest neighbours (NNN) and even further [4]. Frustration gives rise to surprising and subtle effects and often creates very rich phase structures.

To understand better the role played by frustration, some recent Monte Carlo simulations have been made in three and four dimensions with different spin models (Ising, Potts, Heisenberg) described by the following Hamiltonian [5, 6]

$$H = J_1 \sum_{\langle i,j\rangle} \boldsymbol{S}_i \boldsymbol{S}_j + J_2 \sum_{\langle\langle i,l\rangle\rangle} \boldsymbol{S}_i \boldsymbol{S}_l \qquad (1)$$

where $\langle i, j\rangle$ corresponds to NN, and $\langle\langle i, l\rangle\rangle$ to NNN. The phase diagrams so obtained present rather similar phase structures: namely, the addition of a frustrating NNN antiferromagnetic (AF) interaction to a ferromagnetic NN interaction often creates several AF phases, for which the AF order appears first in one dimension, then in two dimensions, and so on, thereby breaking rotational invariance [5].

In this paper, we study analytically the two-dimensional XY model with ferromagnetic NN and antiferromagnetic NNN interactions on a square lattice also defined by (1) (that we can call the ANNNXY model or a two-dimensional $J_1 - J_2$ XY model). Because of the $U(1)$ symmetry, the simple XY model has singular solutions, the vortices, that will disorder

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the system, leading to the famous Kosterlitz–Thouless (KT) phase transition [7–9]. This $J_1 - J_2$ XY model has two distinctive phases, a non-frustrated one with ferromagnetic order and a frustrated one with a particular AF order [10]. We will study the influence of vortices in both phases by means of the real-space renormalization group [8]. The frustrated phase is the most complex one, because spin waves favour a collinear ordering (this constitutes a simple example of 'order by disorder' [11, 10]). An Ising order parameter thus appears dynamically in the model. The symmetry group will be $Z_2 \times U(1)$. A similar situation has already been encountered in the fully frustrated XY model on a square lattice, and in the AF triangular XY model [12, 13]. Both models exhibit specific critical behaviours with an Ising and KT transition at the same point.

The paper will be organized as follows. In section 2, we study the two-dimensional $J_1 - J_2$ XY model with the help of the Villain transformation [14] and translate it into a generalized Coulomb gas language. Because of the Villain approximation, the Ising and KT order parameters are decoupled. In section 3, we derive the generalized KT renormalization group equations. We show in particular, that the usual KT fixed point can be destabilized by frustration in the AF phase. It enables us to propose different scenarii concerning the nature of the phase transition in this ANNNXY model. This analysis suggests that only one transition, essentially dominated by the Ising order parameter, is probable. Finally, in section 4, we give a summary and conclude with speculative remarks concerning possible extensions of these results, in particular, when some disorder is added.

2. Coulomb gas description of the XY model with competing interactions

The two-dimensional $J_1 - J_2$ XY model on a square lattice is defined by the following Hamiltonian

$$H = J_1 \sum_{\langle x,x' \rangle} \cos(\theta_x - \theta'_x) - J_2 \sum_{\langle\langle x,x' \rangle\rangle} \cos(\theta_x - \theta'_x). \tag{2}$$

If $J_2 = 0$ this model describes the usual XY model and leads to the well known KT phase transition induced by vortices [7–9]. When $J_2 \neq 0$, this extended XY model already exhibits a non-trivial ground state at classical level [10, 1]. Let us denote $\eta = J_2/J_1$. When $\eta < \frac{1}{2}$ the ground state is the standard ferromagnetic order, whereas if $\eta > \frac{1}{2}$ the system breaks up into two squares (but this time $(\sqrt{2} \times \sqrt{2})$) sublattices with independent AF order (see figure 1) [10]. The value $\eta = \frac{1}{2}$ is a strong singularity (i.e. a Lifshitz point) where both states are *a priori* possible but also any order states that satisfies $\sum_{\text{plaquette}} \boldsymbol{S}_i = \boldsymbol{0}$. We have the same Lifshitz point with Heisenberg spins.

Let us now study the influence of vortices when temperature is raised. As the two ground states described above are strongly different, we have to compute their excitations independently.

2.1. The non-frustrated phase: $\eta < \frac{1}{2}$

Consider the Hamiltonian (2) in the ferromagnetic phase. By the Villain transformation [14], valid at low temperature,

$$\exp(\beta \cos(\theta - \theta')) \sim C \sum_{m=-\infty}^{+\infty} \exp\left(-\frac{\beta}{2}(\theta - \theta' - 2\pi m\right)^2 \tag{3}$$

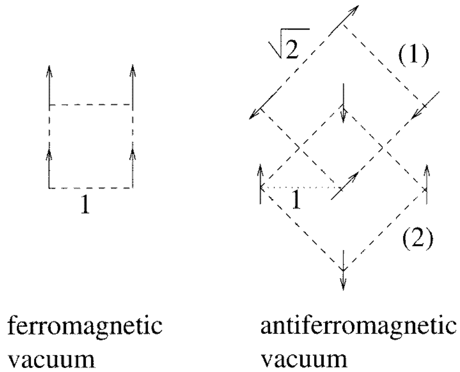

Figure 1. The two different classical vacuum configurations: the usual ferromagnetic one, and the one with two independent antiferromagnetic sublattices. Here we consider an arbitrary angle, ϕ, between the two sublattices.

the partition function can be written as a sum of Gaussians

$$\mathcal{Z} = \left(\prod_{\boldsymbol{x}} \int_{-\pi}^{+\pi} \mathrm{d}\theta_{\boldsymbol{x}}\right) \exp\left(\beta J_1 \sum_{\langle x,x'\rangle} \cos(\theta_{\boldsymbol{x}} - \theta_{\boldsymbol{x}'}) - \beta J_2 \sum_{\langle\langle x,x'\rangle\rangle} \cos(\theta_{\boldsymbol{x}} - \theta_{\boldsymbol{x}'})\right)$$

$$= \left(\prod_{\boldsymbol{x}} \int_{-\pi}^{+\pi} \mathrm{d}\theta_{\boldsymbol{x}}\right) \sum_{\{n_{\boldsymbol{x}}^{\mu}\},\{\tilde{n}_{\boldsymbol{x}}^{\alpha}\}} \exp\left(-\frac{\beta J_1}{2} \mathcal{A}(\theta_{\boldsymbol{x}}, n_{\boldsymbol{x}}^{\mu}, \tilde{n}_{\boldsymbol{x}}^{\alpha})\right) \tag{4}$$

where,

$$\mathcal{A}(\theta_{\boldsymbol{x}}, n_{\boldsymbol{x}}^{\mu}, \tilde{n}_{\boldsymbol{x}}^{\alpha}) = \sum_{\boldsymbol{x},\mu} (\theta_{\boldsymbol{x}+\boldsymbol{e}_\mu} - \theta_{\boldsymbol{x}} - 2\pi n_{\boldsymbol{x}}^{\mu})^2 - \eta \sum_{\boldsymbol{x},\alpha} (\theta_{\boldsymbol{x}+\boldsymbol{e}_\alpha} - \theta_{\boldsymbol{x}} - 2\pi \tilde{n}_{\boldsymbol{x}}^{\alpha})^2. \tag{5}$$

In this expression, we have used a Villain transformation for both cosine terms following one of the prescriptions given in [15]. In these formulae, $\mu = 1, 2$, $\alpha = 1, 2$. $\boldsymbol{e}_\mu$ and $\boldsymbol{e}_\alpha$ indicate respectively the directions $\boldsymbol{i}$, $\boldsymbol{j}$ and $(\boldsymbol{i}+\boldsymbol{j})$, $(\boldsymbol{j}-\boldsymbol{i})$. The term $\{n_{\boldsymbol{x}}^{\mu}\}$, $\{\tilde{n}_{\boldsymbol{x}}^{\alpha}\}$ means we sum in each vertex of the lattice, over four integer link variables (a link is partaken by two vertices, and we have eight links per vertex.

This expression is valid only for $\eta < \frac{1}{2}$ where the classical vacuum is ferromagnetic†, so that the action $\mathcal{A}(\theta_{\boldsymbol{x}}, n_{\boldsymbol{x}}^{\mu}, \tilde{n}_{\boldsymbol{x}}^{\alpha})$ is bounded from below. When we switch off the link variables, we recover the spin-wave Hamiltonian. In fact we could have applied the Villain transformation directly on the spin-wave action (as it is often the case in lattice-gauge theories when one wants to include vortex-like contributions).

There is an apparent problem with the link term $-4\pi^2\eta(\tilde{n}_x^{\alpha})^2$ in the action (5). After the integration over θ_x, the problem no longer survives because the action is bounded from below. We introduce the lattice derivatives $\nabla^{\mu}\theta(x) = \theta(x+\mu) - \theta(x)$ and $\tilde{\nabla}^{\alpha}\theta(x) = \theta(x+\alpha) - \theta(x)$, which are modified in (5) by the gauge-link variables. The action (5) is gauge invariant through

$$\begin{aligned} \theta(x) &\to \theta(x) + 2\pi k(x) \\ n_x^{\mu} &\to n_x^{\mu} - \nabla^{\mu} k(x) \\ \tilde{n}_x^{\alpha} &\to \tilde{n}_x^{\alpha} - \tilde{\nabla}^{\alpha} k(x). \end{aligned} \tag{6}$$

† The domain of validity of the Villain approximation for more complex spin structures is not yet properly established.

Although $\theta(x)$ and n_x^μ, $\tilde{n}_x^\alpha$ appear as independent variables, they are correlated in the partition function because of this gauge invariance. The problem now is to find the dual form of the action, namely to write it in a Coulomb gas form. When the NNN interaction is not present, the vortices defined by $m_x = \epsilon_{\mu\nu}\nabla^\mu n_x^\nu$ ($\epsilon_{\mu\nu}$ being the two-dimensional antisymmetric tensor) play a role analogous to charges (they are also the natural gauge-invariant variables on the dual lattice). When we perform the Gaussian integrals over θ, then $Z = Z_{\mathrm{SW}} * Z_{\mathrm{V}}$ where Z_{SW} represents the spin-wave part

$$Z_{\mathrm{SW}} = \int \mathcal{D}\theta \exp\left[-\frac{\beta}{2}\sum_{x,\mu\alpha}[(\nabla^\mu\theta(x))^2 - \eta(\tilde{\nabla}^\alpha\theta(x))^2]\right] \tag{7}$$

and Z_{V} the vortex part

$$Z_{\mathrm{V}} = \sum_{\{n_x^\mu\}\{\tilde{n}_x^\alpha\}} \exp[-2\pi^2\beta n_x^\mu[\delta^{\mu\nu} - \nabla^\mu\mathcal{P}^{-1}\nabla^\nu]n_x^\nu \\ +4\pi^2\eta n_x^\mu[\nabla^\mu\mathcal{P}^{-1}\tilde{\nabla}^\alpha]\tilde{n}_x^\alpha - 2\pi^2\eta^2\tilde{n}_x^\alpha[\delta^{\alpha\beta} - \tilde{\nabla}^\alpha\mathcal{P}^{-1}\tilde{\nabla}^\beta]\tilde{n}_x^\beta]. \tag{8}$$

$\mathcal{P}^{-1}$ indicates the propagator defined on the lattice by its Fourier transform,

$$(\mathcal{P}^{-1})(\boldsymbol{k}) = \left(\sum_\mu 4\sin^2\frac{k_\mu}{2} - \sum_\alpha 4\eta\sin^2\frac{k_\alpha}{2}\right)^{-1}. \tag{9}$$

The corresponding real-space interaction $\mathcal{V}(x)$ can be computed at a long distance in the usual way (see similar calculations in [16]) and leads to

$$\mathcal{V}(x) \approx \frac{-1}{2\pi}\frac{1}{1-2\eta}\ln\left(4|x|\mathrm{e}^\gamma\sqrt{\frac{1-2\eta}{2}}\right). \tag{10}$$

The propagator is only defined for $\eta < \frac{1}{2}$, i.e. as could have been guessed. It indicates clearly the presence of a new phase for $\eta > \frac{1}{2}$, dominated by antiferromagnetism. The problem now is to write Z_{V} in terms of independent gauge-invariant vortices. After the Gaussian integrations and standard manipulations, the vortex action can be written as,

$$\mathcal{A}(n^\nu, \tilde{n}^\alpha) = \sum_x -2\pi^2\beta(m_1(x)\mathcal{P}^{-1}m_1(x) + \eta^2 m_2(x)\mathcal{P}^{-1}m_2(x) + \eta m_3^{\mu\alpha}(x)\mathcal{P}^{-1}m_3^{\mu\alpha}(x)) \tag{11}$$

with $m_1(x) = \epsilon_{\mu\nu}\nabla^\mu n_x^\nu$ the usual vortex of the XY model, $m_2(x) = \epsilon_{\alpha\beta}\tilde{\nabla}^\alpha\tilde{n}_x^\beta$ a vortex defined on the diagonal sublattices and finally $m_3^{\mu\alpha}(x) = \tilde{\nabla}^\alpha n_x^\mu - \nabla^\mu\tilde{n}_x^\alpha$ corresponds to four vortices built by mixing the two lattices. Globally we have three different geometrical plaquettes representing these vortices: the square 1×1 associated with vortices on the original ferromagnetic lattice, the square $\sqrt{2}\times\sqrt{2}$ associated with vortices on the diagonal sublattices, and $(\sqrt{2}\times 1)$ parallelogram plaquettes associated with vortices on mixed lattices. At each vertex, there are four link variables submitted to a gauge invariance. Once the gauge is fixed ($n_x^1 = 0$ for example) only three independent link variables survive. The ambiguity with the term $-4\pi^2\eta(\tilde{n}_x^\alpha)^2$ in the action has totally disappeared after integration, and the action is now manifestly bounded from below.

The dual Coulomb gas form of the model must have only three point-like, independent vortices. The three vortices, m_i, defined on edges are non-local so it is difficult to check their independence. To solve this problem, we defined three local variables, the triangle plaquettes, which are geometrically independent:

$$\begin{aligned} t_1(x) &= n_x^1 + n_x^2 - \tilde{n}_x^1 \\ t_2(x) &= -n_x^2 - n_{x+j} + \tilde{n}_x^1 \\ t_3(x) &= n_x^1 - n_x^2 - \tilde{n}_{x+i}^2. \end{aligned} \tag{12}$$

In terms of these new variables, the action (11) then reads

$$\begin{aligned} \mathcal{A} &= \sum_{x \neq x'} -2\pi^2 \beta((1-2\eta)^2 (t_1(x) + t_2(x))\mathcal{V}(x-x')(t_1(x') + t_2(x')) + \cdots) \\ &= \sum_{x \neq x'} \pi\beta(1-2\eta) m_1(x) \ln|x-x'| m_1(x') + \cdots \end{aligned} \tag{13}$$

because $m_1(x) = (t_1(x) + t_2(x))$. The ellipses represent here terms like $t_i(x)\nabla^\mu \mathcal{P}^{-1} t_j(x)$ or $t_i(x)\tilde{\nabla}^\alpha \mathcal{P}^{-1} t_j(x)$ with $1 \leqslant i, j \leqslant 3$. These terms behave at large distances as $t_i(x)\frac{1}{|x-x'|}t_j(x')$, and can be neglected in comparison with the logarithm Coulomb interaction (13). They may play a role close to the Lifshitz point, where the spin waves are very soft. In the renormalization-group procedure they are irrelevant in the infrared region. Consequently, we find the usual vortex contribution to the partition function, the inverse temperature is just multiplied by a factor $(1-2\eta)$. This result is not surprising at all: indeed a formally similar derivation could have been done with a NNN ferromagnetic interaction; and we know by universality arguments that the long-wavelength behaviour remains unchanged, so the only excitations which deviate from usual vortices could just have local effects [15]. It is exactly what was shown above but with a NNN antiferromagnetic interaction which does not change the nature of the ground state.

2.2. The frustrated phase: $\eta > \frac{1}{2}$

Fro $\eta > \frac{1}{2}$, the ground state now consists of two square $(\sqrt{2} \times \sqrt{2})$ sublattices, that can be labelled (or coloured) by 1 and 2, each with an AF order whose orientation is defined by Θ_1 and Θ_2 (see figure 1). Because the ground-state energy ($E_0 = -4NJ_2$, where N is the number of sites) is independent of J_1, a non-trivial degeneracy appears in the angle $\Phi = \Theta_1 - \Theta_2$. As was proved by Henley [10], this degeneracy is broken by spin-wave excitations that favour collinear alignment. It is a simple example of ‘order by disorder’ [11]. In that case, rotational invariance is also broken and there is thus AF order in one direction and ferromagnetic order in the orthogonal one. This does not violate the Mermin–Wagner theorem [17] because the continuous degeneracy is replaced by a discrete Ising-like order parameter. Such an ordering was already observed numerically in higher-dimensional models described by equation (1) [6] as was mentioned in the introduction. Whether this is also an ‘order by disorder’ effect remains an open question. Nevertheless, such an analysis takes only spin waves into consideration.

We want now to include the vortices in the action to see their effects. In the following, we consider ϕ, the angle between two NN spins, as a parameter independent of the position. To get a Villain treatment for such a ground state, the usual strategy is to apply the Villain transformation to the spin-wave action [15]. Yet, as two spins separated with a length $2a$ have the same orientations, it will be easier to first map the model on a square lattice (2×2) with ferromagnetic interactions, but now with two spins, 1 and 2, per vertex. We can directly apply the result of Chandra *et al* [2] available in our case. After the gradient expansion of the classical energy, the action on this new lattice can be written as

$$\mathcal{A} = \frac{2J_2}{2T} \sum_x \left[\sum_{i=1,2} (\nabla\theta_i)^2 + 2\lambda(\nabla^x\theta_1\nabla^x\theta_2 - \nabla^y\theta_1\nabla^y\theta_2) \right] \tag{14}$$

where we have defined $\lambda = \frac{J_1 \cos\phi}{2J_2}$.

Notice that, if we do the Gaussian integration over θ_i, we recover the quadratic approximation of the result of Henley [10], namely

$$\mathcal{A}_{\rm SW}(\phi) = \text{constant} - \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \log[2J_2(q_1^2(1-\lambda) + q_2^2(1+\lambda))]$$
$$\sim \text{constant} - 0.32 \left(\frac{J_1 \cos(\phi)}{2J_2} \right)^2. \tag{15}$$

The integration over $\boldsymbol{q}$ shows that spin waves select states with $\cos^2(\phi) = 1$, thus a collinear ordering.

We now have to include the periodicity of θ_i variables in the spin-wave action (14) by applying the Villain transformation on each quadratic term. If we do that directly, we will obtain, after integration over θ_i, a global action representing the spin-waves energy (which behaves as $\cos^2(\phi)$), and a vortex action totally independent of local spin-wave effects. This method does not enable us to see how the Ising order parameter emerges and competes with the KT order parameter. Hence, following Chandra *et al* [2], we include a local quadrupole coupling term

$$\mathcal{A}_c = -\gamma \left(\frac{J_1}{2J_2} \right)^2 \int d^2x (\theta_1 - \theta_2)^2 \tag{16}$$

in the action (14) (with $\gamma = 0.32$) corresponding to local spin-wave effects. The coefficient of this coupling term is defined from (15). After diagonalizing the bilinear form in θ_i, we obtain a massive scalar action plus a massless one, where the 2π periodicity has to be included by means of the Villain transformation. The partition function then reads

$$Z = \int \mathcal{D}\theta_1 \mathcal{D}\theta_2 \sum_{\{n_2^\mu(x), l_2^\mu(x)\}} \exp -[\mathcal{A}_{\rm Ising} + \mathcal{A}_{\rm V}] \tag{17}$$

with

$$\mathcal{A}_{\rm Ising} = \sum_x \frac{J_2}{2T} [(1-\lambda)(\nabla^x\theta_1)^2 + (1+\lambda)(\nabla^y\theta_1)^2] + \gamma \left(\frac{J_1}{2J_2} \right)^2 \theta_1^2 \tag{18}$$

the massive action corresponding to the Ising order parameter, and

$$\mathcal{A}_{\rm V} = \sum_x \frac{J_2}{2T} [(\nabla^\mu\theta_2 - 2\pi n_2^\mu(x))^2 + \lambda[(\nabla^x\theta_2 - 2\pi l_2^x)^2 - (\nabla^y\theta_2 - 2\pi l_2^y)^2]] \tag{19}$$

the massless action, where links variables have been included. In the vortex action (19), we have four links per vertex: $n_2^\mu(x)$ corresponds to antiferromagnetic bonds, and $l_2^\mu(x)$ to ferromagnetic bonds. It is important to have applied the Villain transformation for both terms in the action (19), in order, first, to keep traces of vortices associated to ferromagnetic interactions (1×1 plaquettes in the original lattice), and second, to have a sufficient number of renormalization group (RG) equations to renormalize all coupling constants (T and λ).

Note that we have gauge-invariance conditions similar to (6) in the Villain action (19)

$$\begin{aligned}
&\theta_2(x) \to \theta_2(x) + 2\pi k_2(x) \\
&n_2^\mu(x) \to n_2^\mu(x) - \nabla^\mu k_2(x) \\
&l_2^\mu(x) \to l_2^\mu(x) - \nabla^\mu k_2(x).
\end{aligned} \tag{20}$$

So, at each vertex, three degrees of freedom survive.

The main difficulty is to transform the action (19) into a Coulomb gas one. After some tedious algebra detailed in appendix A, we show that the vortex action is described by

$$\begin{aligned}
\mathcal{A}_{\rm V} = -2\pi^2\beta J_2 \sum_x &(1+\lambda)N(x)\mathcal{P}^{-1}N(x) + \lambda(1-\lambda)L(x)\mathcal{P}^{-1}L(x) \\
&+2\lambda N(x)\mathcal{P}^{-1}L(x) - 2\lambda(N(x)+L(x))\mathcal{P}^{-1}M(x)
\end{aligned} \tag{21}$$

where the propagator, $\mathcal{P}^{-1} = [(\nabla^x)^2(1+\lambda) + (\nabla^y)^2(1-\lambda)]^{-1}$, corresponds in fact to the usual one but on an anisotropic lattice. $N(x) = \epsilon^{\mu\nu}\nabla^\mu n_2^\nu$ are the vortices on the AF diagonal sublattices ($\sqrt{2}\times\sqrt{2}$ plaquettes), $L(x) = \epsilon^{\mu\nu}\nabla^\mu l_2^\nu$, the vortices on the normal ferromagnetic lattice (1×1 plaquettes), and $M(x) = \nabla^1 n_2^2 - \nabla^2 l_2^1$, the vortices on the mixed sublattices ($1\times\sqrt{2}$ plaquettes). The vortex action (19) is thus transformed on a Coulomb gas action in anisotropic space with three charge species interacting with each other. The charge is conserved inside each species. Notice that the resulting action is far more complicated than the one describing the non-frustrated phase as expected. Of course, such a treatment decouples both order parameters. We will discuss the validity of this approximation in the next section. Let us first extend the KT renormalization group equations to the actions (13) and (21).

3. Generalized KT equations and phase structures

There are many ways to find the RG trajectories for the standard XY model. First, one can directly use the language of Coulomb gas by introducing a dielectric constant and the polarizability of the dipole pair (see for example [18]). The second way is to notice that the XY model with external fields is dual to the sine–Gordon model, and then to use diagrammatic expansions. Because of its standard perturbative aspect, this method enables us to compute higher-order corrections to the KT equations [19]. Thirdly, a direct real-space renormalization group according to Kosterlitz can be done by arguing that neighbouring vortices with opposite charges have only a short-range effect [8].

3.1. Extension of the KT equations

In this paper, we follow this third method. For the non-frustrated phase, the partition function yields using (10) and (13):

$$Z(\beta,\eta,z_1) = Z_{\rm SW}\sum_{\{m_1(x)\}}\exp\left[\pi\beta(1-2\eta)\sum_{x\neq x'} m_1(x)m_1(x')\log\frac{|x-x'|}{a} + \log z_1\sum_x m_1(x)^2\right] \tag{22}$$

where a is the lattice constant and z_1 the fugacity one adds to control the vortex number. We have seen that the ferromagnetic phase has exactly the same form as the usual XY model with a temperature $\tilde\beta = (1-2\eta)\beta$. In this phase, the ratio $\eta = \frac{J_2}{J_1}$ does not renormalize separately, because it is coupled to totally irrelevant terms (see equation (13)) and so can be considered as a parameter.

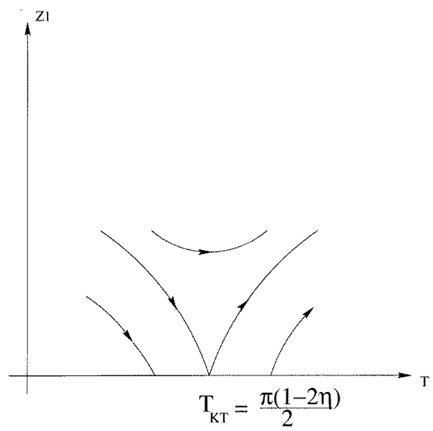

Figure 2. The RG flow for the non-frustrated phase for an arbitrary value of the parameter η. We have the usual flow of the XY model except that $T_{\rm KT}$ is scaled by a factor $(1-2\eta)$.

The RG equations are then the usual ones:

$$\begin{aligned}\frac{\mathrm{d}\tilde\beta}{\mathrm{d}l} &= -4\pi^3\tilde\beta^2 z_1^2\\ \frac{\mathrm{d}z_1}{\mathrm{d}l} &= -z_1(\pi\tilde\beta-2).\end{aligned} \tag{23}$$

The flow for z_1 is represented in figure 2 and corresponds to the usual KT one for a given η. When $\eta\to\frac{1}{2}$, the KT temperature $T_{\rm KT}\to 0$, meaning that vortices tend to proliferate already at very low temperature. This result is not surprising at all, because at the Lifshitz point, every configuration verifying $\sum_{\rm plaquette}\boldsymbol{S}_i=\boldsymbol{0}$ is allowed, so in particular vortices.

Let us now concentrate on the frustrated phase. We have seen that the action decomposes into a massive part (18) plus a Coulomb gas part (21). Let us first consider this non-trivial part. We can transform the vortex action in the frustrated phase on a Coulomb gas with three species can be encountered when we study the XY model with random phase-shifts ($H=-J\sum_{\langle r,r'\rangle}\cos(\theta_r-\theta_{r'}-A_{\langle r,r'\rangle})$). The standard method used to take into consideration the randomness (represented here by the random field $A_{\langle r,r'\rangle}$) is to replicate p times the Coulomb gas [20] and to generalize the KT equations using the appropriate language for Coulomb gas. Let us consider the case of p replicas of the Coulomb gas described by

$$\mathcal{A}_{\mathcal{V}}=\sum_{r\neq r'}\sum_{i,j}m_i(\boldsymbol{r})\mathcal{C}_{ij}m_j(\boldsymbol{r}')\ln\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{a}+\sum_{\boldsymbol{r},i}\log z_i m_i(\boldsymbol{r})^2 \tag{24}$$

where $|\boldsymbol{r}|^2=\frac{x^2}{1-\lambda}+\frac{y^2}{1+\lambda}$ because of anisotropy, and $\mathcal{C}$ is the matrix $(p\times p)$ of coupling constants. As usual, we will only take charge of modulus one, but we will keep the $|m|$ as variables for notations. As was first noticed by Korshunov [20], the interaction between different replicas is of great importance because it hides a part of disorder. In our case, we also have this intervortex species coupling, which is not present in the ferromagnetic phase. This may also indicate a disordering of our initial ground state by all these vortices excitations. The generalization of the KT equations for this model is given in the appendix B using real-space renormalization. The result is as follows:

$$\frac{\mathrm{d}\mathcal{C}_{ij}}{\mathrm{d}l} = -\frac{(2\pi)^2}{\sqrt{1-\lambda^2}} m^2 \sum_k (z_k)^2 \mathcal{C}_{ik}\mathcal{C}_{kj}$$
$$\frac{\mathrm{d}z_k}{\mathrm{d}l} = (2 - \mathcal{C}_{kk} m^2) z_k \tag{25}$$

where m^2 is equal to $\frac{\pi}{2}$ here. Our case corresponds to a 3×3 matrix, C_{ij}, whose initial conditions are defined from equation (21). z_1 is the fugacity associated to N vortices, z_2 to L vortices and finally z_3 to M vortices defined in (21).

Note that the terms, coupling different vortex types, can provide fugacities associated to hybrid vortices (configurations in which vortex of different species reside at the same site) [21]. We have three possible hybrid vortices in our case. For simplicity and readability, we have not presented the more general study which incorporates these three hybrid fugacities. The KT equations can be generalized in a straightforward way following [21]. We have checked that this does not alter the forthcoming results, but on the contrary enforces them.

3.2. Discussion of a possible phase diagram

Before discussing a possible phase diagram, we will study carefully the flow associated with equation (25) based only on the KT order parameter.

Let us first see that we recover the singularity ($\lambda = 1$) as in the non-frustrated case but this time in the denominator. Hence, the two ground states do not have the same behaviour when the singularity is approached. This is not so surprising because their symmetry is different. This has already been noticed and studied with quantum fluctuations with Heisenberg antiferromagnets in [1]. In the first block of equations, we have six variables obeying initial conditions (see equation (25)) despite our model having only two parameters (β and λ). In fact, initially C_{33}, the inverse temperature associated to type 3 vortices, equals zero, so because of the nature of the equation governing its behaviour, C_{33} remains zero. Taking care of the initial conditions (we work at low fugacities), we can study numerically the flow of renormalization coupling constants (it is in fact much more convenient to use the inverse of the C_{ij}).

We will consider the generic case $J_2 = J_1 = 1$. We will especially follow the fugacities with temperature and compare it to the usual KT flow (the physical temperature corresponds to C_{11}^{-1} in order to recover the KT flow when $\lambda = 0$). The associated fugacity, z_1, corresponds to vortices on the antiferromagnetic lattices. In figure 3, we have represented the fugacity z_1 function of the temperature T. We see that for low-temperature and low-fugacity initial conditions we do not recover the usual expected KT fixed point despite the strong cross-over regime! We have checked numerically that this result is independent of initial conditions, namely it is valuable even at very low fugacity. How shall we interpret such a puzzling result? In figure 4 we have represented the evolution of fugacities z_2 and z_3 with T (z_2 is the fugacity associated to ferromagnetic (F) 1×1 vortices and z_3 to mixed (F–AF) $1 \times \sqrt{2}$ vortices). They are driven in a high-fugacity domain. This can be directly correlated to the evolution of their associated temperature C_{22}^{-1} and C_{33}^{-1}. Indeed, in figure 5, the coupling constants, C_{ij} are represented. We observe that C_{22} goes towards zero, whereas C_{33} remains zero (see above), meaning that the system is driven in a strong coupling regime for vortices of type 2 and 3. The most important thing is that the coupling constant, C_{13}, remains finite (it couples the type 1 vortices to the type 3 vortices). Hence, the disorder caused by type 2 and 3 vortices is strong enough to eliminate the usual KT fixed point. To check this, in figure 6 we have shown $z_1(T)$ and $z_2(T)$ at $z_3 = 0$. The KT fixed point is

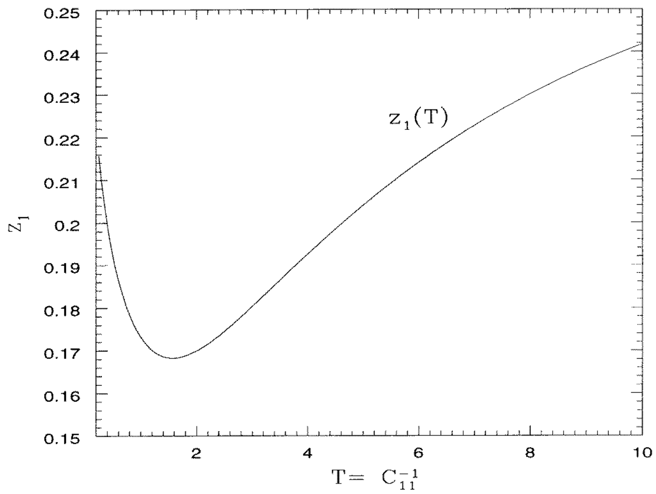

Figure 3. The RG system (25) is solved at $J_2 = J_1 = 1$ and low fugacities. We have represented the evolution of the fugacity, z_1, with temperature, T, defined by $T = C_{11}^{-1}$. The main feature is that the KT fixed point is lost.

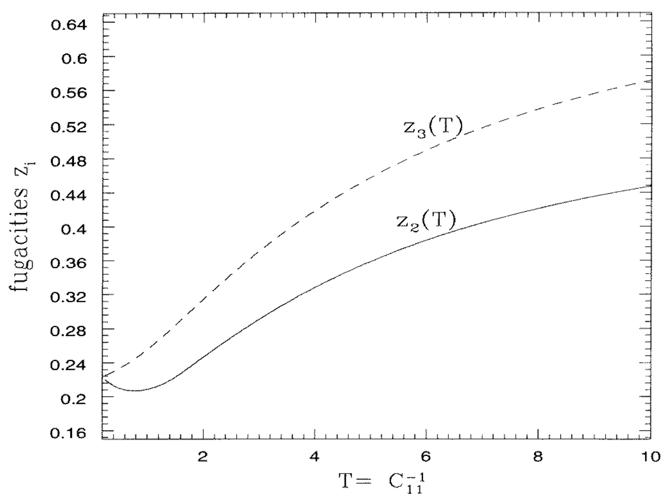

Figure 4. The full curve represents the evolution of the fugacity, z_2, with the temperature $T = C_{11}^{-1}$ whereas the broken curve represents the fugacity, z_3. They are driven to a high-density regime that will disorder the usual KT fixed point.

then recovered. Hence, the KT fixed point is suppressed essentially by type 3 vortices. This can be seen geometrically: when we built a vortex around a 1×1 plaquette, the AF order on diagonal sublattices is not roughly affected, contrary to the case of $1 \times \sqrt{2}$ plaquettes. As $\lambda \sim \frac{C_{22}}{C_{11}} \to 0$ in the infrared limit (see figure 5), the two sublattices try to decouple each other as already observed in the case for Heisenberg spins [2]. When both sublattices are weakly coupled, it becomes easier (geometrically and energetically) to form a vortex of type 2 or 3 from the collinear ground state. So, it is not surprising that such vortices are led in a high-density regime when $\lambda \to 0$. However, the fact that it is sufficient to destabilize the KT fixed point for both sublattices is unexpected.

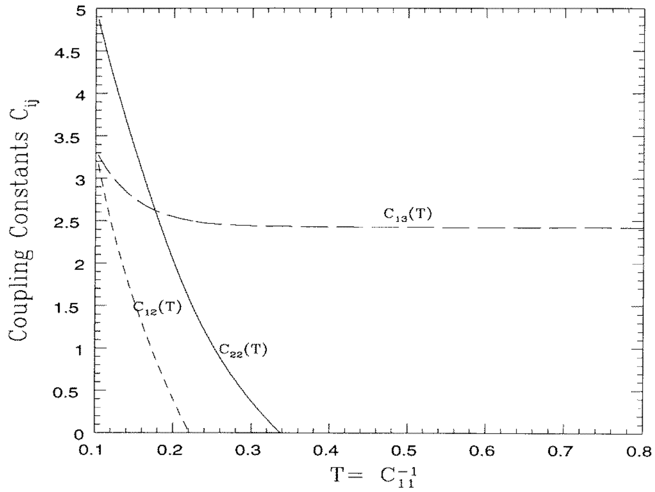

Figure 5. The evolution of some coupling constants C_{ij}. The full curve represents the evolution of C_{22} with the temperature $T = C_{11}^{-1}$. The fact that $C_{22} \to 0$ can be interpreted as the decoupling of both AF sublattices. But, this also indicates a strong coupling regime associated to C_{22}^{-1}, the 'temperature' associated to type 2 vortices. The short broken curve represents the evolution of C_{12} with T. C_{12} corresponds to the coupling between type 2 and type 1 vortices. Finally, the long broken curve represents the evolution of C_{13} with T. C_{13} converges toward a finite value explaining why type 3 vortices are able to disorder the expected KT fixed point for type 1 vortices. The curve $C_{23}(T)$ is rather similar than $C_{13}(T)$ so has not been represented.

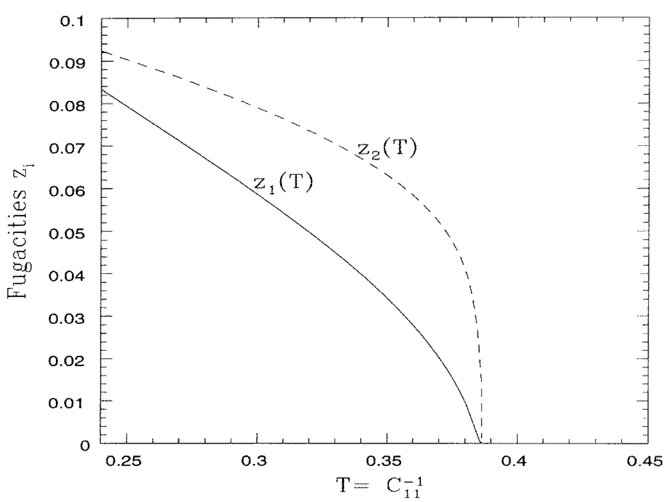

Figure 6. We have now enforced the fugacity, z_3, to stay at 0. The full curve represents the evolution of z_1 with T, whereas the broken curve corresponds to the evolution of z_2 with T. The KT fixed point is recovered. This plot shows the aim of the role played by type 3 vortices.

We see in figure 3 that we have a strong cross-over regime associated to the fugacity z_1 as long as $T < \frac{\pi J_2}{2}$. This could play a role as we will see later. For $T > \frac{\pi J_2}{2}$ we recover the usual UV fixed line, proving that the field theory has meaning.

We now have a consistent interpretation of the flow (figures 3–6) where only the role of vortices was discussed. We must now take both order parameters into account for a global analysis of this $J_1 - J_2$ model. Indeed, equation (18) describes a massive scalar theory associated with the Ising transition. We notice that when $\frac{J_2}{J_1}$ increases, the mass proportional to $(\frac{J_1}{2J_2})^2$ decreases meaning that the critical temperature associated with this Ising transition reaches its maximum value when $J_2 = \frac{J_1}{2}$. This is already the case for Heisenberg spins, where only the Ising order parameter is present [2]. So equation (18), based on spin-waves analysis, predicts a low-temperature phase with a global collinear ordering. In this approach, we have decoupled both order parameters. It is clear that they are correlated; the numerous numerical and analytical works on the fully frustrated XY model on a square lattice, the AF triangular XY model and so on, which also have a $Z_2 \times U(1)$ symmetry, have proved this point [12, 13]. It has to be noted that in this $J_1 - J_2$ XY model, the Ising order parameter emerges in a dynamical way (by 'order by disorder' effect) contrary to the models quoted above. So, it is not at all obvious to compare it and to directly conclude that the $J_1 - J_2$ model must have a critical line of Ising–KT type (as it is often the case for such models). Nevertheless, the analysis made in this paper enables us to eliminate some *a priori* possible scenarii.

Suppose we have two critical transitions, T_{KT} and T_{Ising}. The scenario $T_{\mathrm{KT}} < T_{\mathrm{Ising}}$ is impossible because we have seen that the usual KT fixed point is destabilized. In that case we would have no transition at all, only a disordered ground state even at low temperature. It would contradict the results of classical (and quantum) spin waves, that are usually valuable at very low temperature. Moreover, Henley [10] has found numerically an ordered collinear phase at low T.

On the contrary, if an Ising transition occurs first (this supposes we have a collinear ordered phase at low T), the ground state at $T \geqslant T_{\mathrm{Ising}}$ would consist of domain walls with global orientation $\cos(\phi) = \pm 1$. If the associated Ising correlation length (the size of a domain wall) is large enough, we could apply the result to equation (25) inside each domain wall. Hence, vortices of type 2 and 3 are able to suppress the KT fixed point, so only one fixed point of Ising type separating a collinear and a disordered phase occurs. Yet, if the correlation length is not large enough, then the cross-over associated with type 1 vortices could play an important role. In that case, we recover approximately the KT low-temperature behaviour for a type 1 vortex, namely they tend to bind at this scale. So it is not unusual to have a KT transition at $T > T_{\mathrm{Ising}}$. Nevertheless, the study of models with $Z_2 \times U(1)$ order parameters tend to show (it is not yet completely clear) that the Ising transition could 'trigger' the KT transition.

Different mechanisms have been proposed, notably a screening effect caused by domain walls [13]. In our case, we would have to include these domain-wall effects plus the disorder implied by type 2 and 3 vortices. It is plausible to think that all these effects will enforce the unboundness of type 1 vortices just after the Ising transition has occurred. In that case, the staggered magnetization would have the sign of KT behaviour and we would also find one fixed point of Ising–KT type. The present analysis does not enable us to decide between these last two possibilities (Ising or Ising–KT). It seems that the Ising order parameter plays the most important role that could be checked by accurate numerical simulations. As very few simulations cover this model, it is difficult to be conclusive. Henley [10] has made a study of this model in the presence of dilution (that selects anticollinear ordering). They predict for the case $J_2 = J_1 = 1$ only one transition at $T = 0.97$ separating a collinear and a disordered ground state. The precise nature of this transition was not discussed numerically. This result enforces both possible scenarii proposed above. In that case, we would obtain in the $(T, \frac{J_2}{J_1})$ plane the schematic phase diagram presented in figure 7. In the non-frustrated phase we have a line of KT type separating a ferromagnetic phase by a disordered one as we have seen above. Whereas in the frustrated phase, we have a curve separating a

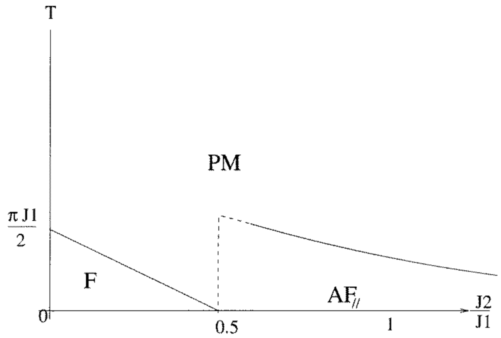

Figure 7. A schematic phase diagram in the plane $(T, \frac{J_2}{J_1})$. The nature of the curve separating the collinear order phase from the disordered one remains of Ising or Ising+KT type. The results have been extrapolated close to the Lifshitz point where this analysis fails.

collinear order from a disorder phase. So, the exact nature of this transition remains to be investigated in more detail by another approach enabling the coupling of the Ising and KT order parameters.

4. Conclusion and outlook

In this paper, we have studied an XY model with ferromagnetic NN interactions and AF NNN interactions. In the ferromagnetic phase, we show that the only relevant vortices are the usual ones, leading to a KT transition at $T_{\mathrm{KT}} = \frac{\pi J_1}{2}(1 - \frac{2J_2}{J_1})$. The frustrated phase is much richer. By 'order by disorder' effect, an Ising order parameter associated to a collinear ground state is generated dynamically. Hence, the symmetry group becomes $Z_2 \times U(1)$. We have treated both order parameters in an independent way. The greater part of the paper was devoted to a generalization of the KT equations in the collinear phase. Three types of vortices were identified. We have thus found that the expected KT fixed point associated to one type of vortex is suppressed due to the presence of the other vortices, which are driven to a high-fugacity regime. Nevertheless, a strong cross over was marked. The analysis, based on the Ising order parameter, forecasts a low-temperature phase dominated by a collinear alignment. By comparing both approaches, it seems that only two scenarii are possible: an Ising transition line or an Ising–KT transition separating the collinear from the disordered phase. The nature of the transition requires a different approach able to take into consideration the coupling between both order parameters. A possibility is to include a coupling between spin waves and vortices. Some attempts in this direction have been done by Benakli *et al* [22] for another model. It would also be interesting to include some disorder effects in this model, notably by adding random-phase shifts. With the non-frustrated phase being of the same universality class as the usual XY model, it is reasonable to think that the results of [20] would apply in the ferromagnetic phase. Yet, in the frustrated phase, the question is far from being trivial. We can show that disorder, like dilution[10], favours an anticollinear order, so it offers the possibility to suppress the Ising order parameter for high disorder even at low temperature. This will be a subject of future work.

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Appendix A

In this appendix, we transform the Villain action $\mathcal{A}_{\rm V}$ defined in (19) into a Coulomb gas one. The action then reads

$$\mathcal{A}_{\rm V} = -\frac{\beta J_2}{2}\sum_x (\nabla^\mu\theta_2 - 2\pi n_2^\mu(x))^2 + \lambda((\nabla^x\theta_2)^2 - (\nabla^y\theta_2)^2) \\ +4\pi\lambda[(\nabla^x l_2^1) - (\nabla^y l_2^2)] + 4\pi^2\lambda[(l_2^1)^2 - (l_2^2)^2] \quad \text{(A1)}$$

The integration over θ_2 is now easy,

$$\mathcal{A}_{\rm V} = 2\pi^2\beta J_2\sum_x [\nabla^\mu n_2^\mu(x) + \lambda(\nabla^x l_2^1 - \nabla^y l_2^2)]\mathcal{P}^{-1}[\ldots] \\ -2\pi^2\beta J_2(n_2^\mu)^2 - 2\pi^2\beta J_2\lambda((l_2^1)^2 - (l_2^2)^2) \quad \text{(A2)}$$

with $\mathcal{P}^{-1} = [(\nabla^x)^2(1+\lambda) + (\nabla^y)^2(1-\lambda)]^{-1}$, corresponding to an isotropic propagator. Then we develop and introduce vortex variables ($N(x) = \epsilon^{\mu\nu}\nabla^\mu n_2^\nu$ and $L(x) = \epsilon^{\mu\nu}\nabla^\mu l_2^\nu$),

$$\mathcal{A}_{\rm V} = -2\pi^2\beta J_2\sum_x N(x)\mathcal{P}^{-1}N(x) - \lambda^2 L(x)\mathcal{P}^{-1}L(x) \\ +\lambda[(\nabla^\mu l_2^1 - \nabla^x n_2^\mu)\mathcal{P}^{-1}(\nabla^\mu l_2^1 - \nabla^x n_2^\mu)] \\ -\lambda[(\nabla^\mu l_2^2 - \nabla^y n_2^\mu)\mathcal{P}^{-1}(\nabla^\mu l_2^2 - \nabla^y n_2^\mu)]. \quad \text{(A3)}$$

The last term describes local and non-local vortices that are a mixture of n and l bounds. Let us introduce $M(x) = \nabla^1 n_2^2 - \nabla^2 l_2^1$ and also

$$M'(x) = \nabla^1 l_2^2 - \nabla^2 n_2^1 = L(x) + N(x) - M(x)$$

the local vortices built with l links (so associated with the ferromagnetic interaction originally) and n links AF interactions. We can write the last two terms of the action with $M(x)$ and $M'(x)$, the non-local vortices having only short-range interactions. This can be seen more rigorously (following the same treatment as in section 3.1) if we introduce adapted triangular vortices and use the gauge conditions $l_2^1(x) = l_2^2(x)$. It leads to

$$\mathcal{A}_{\rm V} = -2\pi^2\beta J_2\sum_x N(x)\mathcal{P}^{-1}N(x) - \lambda^2 L(x)\mathcal{P}^{-1}L(x) - \lambda M(x)\mathcal{P}^{-1}M(x) \\ +\lambda M'(x)\mathcal{P}^{-1}M'(x). \quad \text{(A4)}$$

Since $M'(x) = L(x) + N(x) - M(x)$, we finally obtain the action (21).

Appendix B

In this appendix we want to renormalize, in real anisotropic space, a replicated Coulomb gas. Let us consider the grand-partition function associated with equation (25),

$$\mathcal{Z} = \sum_{n_1,\ldots,n_p}\frac{z_1^{2n_1}\ldots z_p^{2n_p}}{(n_1!)^2\ldots(n_p!)^2}\int_{D_{2np}^p}\cdots\int_{D_{2n_1}^1}{\rm d}^2 r_{2n_1,1}\ldots\int_{D_{\alpha_1}^1}{\rm d}^2 r_{\alpha_1}\ldots\int_{D_1^1}{\rm d}^2 r_{1,1} \\ \times\exp\left[\sum_{i,j}\sum_{\alpha_i,\beta_j} m_{\alpha_i}\mathcal{C}_{ij}m_{\beta_j}\log\frac{|r_{\alpha_i} - r_{\beta_j}|}{a}\right]. \quad \text{(B1)}$$

In this expression, we have generalized the notations of Kosterlitz [8]. We have used also the norm $|\boldsymbol{r}|^2 = \frac{x^2}{1-\lambda} + \frac{y^2}{1+\lambda}$ which takes the anisotropy into account. The order of the integration is fixed and $D^k_{\alpha_k}$ is the whole plane except for the ellipses $|r_{\alpha_k} - r_{\beta_l}| < a$, with $l > k$, or $l = k$ and $\beta_l > \alpha_l$. To fix notations, we write m_{α_i} for the αth vorticity of the ith replica (here we take only vortex charges ± 1, hence with the normalizations it gives $m^2 = \frac{\pi}{2}$), with Latin indices standing for replica and Greek indices for vortices inside a replica. As in the Kosterlitz derivation, the final result will not depend on the ordering. The idea is now to scale the lattice spacing for a to $a + \mathrm{d}a$ and to integrate explicitly the contribution from vortex–antivortex pairs (i.e. with $m_{\alpha_k} = -m_{\beta_k}$) in the same replica whose relative coordinates lie within an elliptical annulus of radius $\mathrm{d}a$. Now we have just to follow the same procedure as the Kosterlitz one inside each replica. We rearrange the integral as follows

$$\int_{D^p_{2n_p}} \cdots \int_{D^1_1} \mathrm{d}^2 r_{1,1} = \int_{D'^p_{2n_p}} \cdots \int_{D'^1_1} \mathrm{d}^2 r_{1,1} + \frac{1}{2} \sum_k \sum_{\alpha_k \neq \beta_k} \int_{D'^p_{2n_p}} \cdots \int_{D'^k_{2n_k}} \cdots \int_{D'^k_{\alpha_k+1}} \times \int_{D'^k_{\alpha_k-1}} \cdots \int_{D'^k_{\beta_k+1}} \int_{D'^k_{\beta_k-1}} \cdots \int_{D'^1_1} \mathrm{d}^2 r_{1_1} \int_{\bar{D}^k_{\alpha_k}} \mathrm{d}^2 r_{\alpha_k} \int_{\delta^k_{\beta_k}(\alpha_k)} \mathrm{d}^2 r_{\beta_k} \tag{B2}$$

where D' is defined as for D with $\alpha \to a + \mathrm{d}a$, $\bar{D}^k_{\alpha_k}$ the plane without elliptical disk around all other points and $\delta^k_{\beta_k}(\alpha_k)$ the annulus defined by $a < |r_{\alpha_k} - r_{\beta_k}| < a + \mathrm{d}a$. So we have to compute

$$\int_{\delta^k_{\beta_k}(\alpha_k)} \mathrm{d}^2 r_{\beta_k} \exp\left(2 \sum_{l,\gamma_l} \mathcal{C}_{lk} \left[m_{\alpha_k} m_{\gamma l} \log \frac{|r_{\alpha_k} - r_{\gamma l}|}{a} + m_{\beta_k} m_{\gamma l} \log \frac{|r_{\beta_k} - r_{\gamma l}|}{a} \right] \right) \tag{B3}$$

paying attention to the anisotropy.

We put $\boldsymbol{\rho} = \boldsymbol{r}_{\alpha_k} - \boldsymbol{r}_{\beta_k}$, then $|\boldsymbol{\rho}| \sim a$ according to our norm. If we require $m_{\alpha_k} = -m_{\gamma l}$, we have to calculate

$$\int_{\delta^k_{\beta_k}(\alpha_k)} \mathrm{d}^2 r_{\beta_k} \prod_{l,\gamma l} \left(\left[1 + \frac{2\boldsymbol{\rho} \cdot (\boldsymbol{r}_{\alpha_k} - \boldsymbol{r}_{\gamma l})}{|\boldsymbol{r}_{\alpha_k} - \boldsymbol{r}_{\gamma l}|^2} + \frac{a^2}{|\boldsymbol{r}_{\alpha_k} - \boldsymbol{r}_{\gamma l}|^2} \right]^{\mathcal{C}_{kl} m_{\alpha_k} m_{\gamma l}} \right). \tag{B4}$$

The integrations are now formally identical with those of Kosterlitz (the difference lies within the scalar product associated to the norm so will not affect the coupling constant) except the surface element which reads $\frac{2\pi a\, \mathrm{d}a}{\sqrt{1-\lambda^2}}$. Using the derivation of Kosterlitz the grand-partition function then yields:

$$\mathcal{Z} = \sum_{n_1,\ldots,n_p} \frac{z_1^{2n_1} \ldots z_p^{2n_p}}{(n_1!)^2 \ldots (n_p!)^2} \int_{D'^p_{2n_p}} \cdots \int_{D'^1_{2n_1}} \mathrm{d}^2 r_{2n_1,1} \ldots \int_{D'^1_1} \mathrm{d}^2 r_{1,1} \times \left[1 + \sum_k z_k^2 \frac{2\pi a\, \mathrm{d}a}{\sqrt{1-\lambda^2}} \left(A - 2\pi a^2 \sum_{l,l'} \mathcal{C}_{kl} \mathcal{C}_{kl'} m^2 \sum_{\beta_l \neq \beta_{l'}} m_{\beta_l} m_{\beta'_l} \log \frac{|r_{\beta_l} - r_{\beta'_l}|}{a} \right) \right] \times \exp\left[\sum_{i,j} \sum_{\alpha_i,\beta_j} m_{\alpha_i} \mathcal{C}_{ij} m_{\beta_j} \log \frac{|r_{\alpha_i} - r_{\beta_j}|}{a} \right]. \tag{B5}$$

By using the fact that we work with $\mathrm{d}a \ll a$ we can exponentiate the second line and see that the coupling constant renormalize as

$$\mathcal{C}_{ll'} \to \mathcal{C}_{ll'} - \frac{(2\pi)^2}{\sqrt{1-\lambda^2}} m^2 \frac{\mathrm{d}a}{a} \sum_k (z_k)^2 \mathcal{C}_{lk} \mathcal{C}_{kl'}. \tag{B6}$$

It still remains to renormalize the fugacity. We use

$$\log \frac{|r_{\alpha_i} - r_{\beta_j}|}{a} \sim \log \frac{|r_{\alpha_i} - r_{\beta_j}|}{a + \mathrm{d}a} + \frac{\mathrm{d}a}{a}.$$

Because of charge conservation inside each replica, only the coupling constant of the type $\mathcal{C}_{ll}$ will play a role as in the standard case. So, we have the usual renormalization equation for z_l

$$z_l \to \left(2 - \mathcal{C}_{ll} m^2 \frac{\mathrm{d}a}{a}\right) z_l. \tag{B7}$$

The equations (B6) and (B7), describing the RG flow, have the same form as the one obtained by Scheidl [20].

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